A numerical approach for Camassa–Holm equation using extended trial equation method

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Abstract

In this article, we attempt to solve the Camassa–Holm equation using extended trial equation method (ETEM). The validity and applicability of the method are operated for two different cases of the relationship equation obtained as a result of balance principle. Finally, we obtain single kink and hyperbolic function solution of the problem.

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1 Introduction

Scientific modeling of nonlinear phenomena in the real world is quite common. Nonlinear partial differential equations (NPDEs) representing a variety of mathematical models in real–life applications arise in many different fields, such as fluids and quantum mechanics, ecology, physical systems, thermodynamics, chemistry, engineering, mathematics, and biology, etc. It is crucial important to make a decison of which strategy should be adopted to analyze the behaviour of equation. The solution of these equations can include some obstacles such as irregularity or uniqueness. Within this framework, associated with our work, we briefly give some numerical methods such as G'/G-expansion method [20], homotopy analysis method [7], finite difference methods [17], finite element method [13], multiple exp-function method [14], homotopy perturbation method [16], sub-ODE method [18], variational iteration method [8], modified extended tanh–function method [10], spectral collocation and waveform relaxation methods [12], Exp–Function Method [15], the extended F-expansion method [1], and Laguerre Pseudospectral method [19].

The propose of this paper is to investigate hyperbolic function and single kink solution of Camassa–Holm equation by employing ETEM. This method has often been used by many authors in a variety of differential equations. For example, Gebreel [11] get some different kinds of exact solutions to nonlinear coupled Schrodinger–Boussinesq equations. Ekici et al. [9] use the present method and G'/G-expansion to obtain solitons of the Kundu–Eckhaus equation. Recently, Biswas et al. [2, 3, 4, 5] obtain optical solitons by carrying out ETEM to some different model problems.

This paper is arranged as follows. In Section 2, we introduce ETEM for a given general NPDEs. In Section 3, we apply proposed method to Camassa-Holm equation. Finally, the conclusion is drawn in Section 4.

2 Extended trial equation method

In this section, we define the ETEM based on a general nonlinear partial differential equation.

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Step 1. Consider a nonlinear PDE as follows

$$P(y, y_t, y_x, y_{xx}, y_{tt}, ...) = 0. \tag{2.1}$$

Firstly, we employ the following general wave transformation formula

$$y(x,t) = y(\eta), \eta = x - wt, \tag{2.2}$$

where $w \neq 0$. Thus, the following nonlinear ODE is constructed by substituting (2.2) into (2.1)

$$N(y, y', y'', ...) = 0. \tag{2.3}$$

Step 2. Secondly, we describe the trial equation as

$$y = \sum_{j=0}^{\delta} \tau_j \kappa^j, \tag{2.4}$$

where

$$(\kappa')^2 = \Lambda(\kappa) = \frac{\Phi(\kappa)}{\Psi(\kappa)} = \frac{\xi_\theta \kappa^\theta + \ . \ . \ . \ + \xi_1 \kappa + \xi_0}{\zeta_\in \kappa^\in + \ . \ . \ . \ + \zeta_1 \kappa + \zeta_0}. \tag{2.5}$$

Taking into consideration (2.4) and (2.5), we obtain

$$(y')^2 = \frac{\Phi(\kappa)}{\Psi(\kappa)} \left(\sum_{j=0}^{\delta} j \tau_j \kappa^{j-1} \right)^2, \tag{2.6}$$

$$y'' = \frac{\Phi'(\kappa)\Psi(\kappa) - \Phi(\kappa)\Psi'(\kappa)}{2\Psi^2(\kappa)} \left(\sum_{j=0}^{\delta} j \tau_j \kappa^{j-1} \right) + \frac{\Phi(\kappa)}{\Psi(\kappa)} \left(\sum_{j=0}^{\delta} j(j-1) \tau_j \kappa^{j-2} \right), \tag{2.7}$$

where $\Phi(\kappa)$ and $\Psi(\kappa)$ are two polynomials. Substituting (2.4), (2.6), and (2.7) into (2.3), we provide a polynomial equation $\Omega(\kappa)$ of κ as follows

$$\Omega(\kappa) = \mu_s \kappa^s + \ . \ . \ . \ + \mu_1 \kappa + \mu_0 = 0. \tag{2.8}$$

We then find a relationship between the unknown basic parameters θ, $\in$, and δ by employing balance principle.

Step 3. Consider the coefficients of $\Omega(\kappa)$ are equal to zero. We take a systems of algebraic equations as follows

$$\mu_j = 0, \ j = 0, \ . \ . \ . \ , s. \tag{2.9}$$

By solving (2.9), we obtain the values of $\xi_0, \ . \ . \ . \ , \xi_\theta$; $\zeta_0, \ . \ . \ . \ , \zeta_\in$, and $\tau_0, \ . \ . \ . \ , \tau_\delta$.

Step 4. Simplify (2.5) to the integral form, we get

$$\pm(\eta - \eta_0) = \int \frac{\mathrm{d}\kappa}{\sqrt{\Lambda(\kappa)}} = \int \sqrt{\frac{\Psi(\kappa)}{\Phi(\kappa)}} \mathrm{d}\kappa \tag{2.10}$$

Implementing a complete discrimination systems of polynomial to categorize the roots of $\Phi(\kappa)$ and solving (2.10), we obtain the exact solutions to nonlinear PDE.

3 Implementing of ETEM to Camassa-Holm equation

In this section, we consider the generalized nonlinear Camassa–Holm equation as follows:

$$y_t + 2ky_x - y_{xxt} + (b+1)yy_x = by_x y_{xx} + yy_{xxx}, \tag{3.1}$$

where, for $b = 2$, the function $y(x,t)$ is defined as the fluid velocity for $x \in \mathbb{R}$, $t \geq 0$, and $k \geq 0$. Equation (3.1) is defined as a model for determining shallow water gravity waves with higher order dispersion [6]. It also has the conservation laws and can be integrable. If we choose $k = 0$, then the Camassa-Holm equation reach the peakon solutions that has solitons with a sharp and discontinuity peak in wave slope. Now, we will apply ETEM to solve Camassa-Holm equation a step by step approach.

At first, we take the transformation

$$y(x,t) = y(\eta),\ \eta = x - wt,$$

where w is considered as an arbitrary constant. Integrating the resulting equation for η and constituting the integration constant to zero, we obtain the following nonlinear ODE

$$(-w+y)y'' + \frac{(y')^2}{2} - (-w+2k)y - 3\frac{y^2}{2} = 0. \tag{3.2}$$

Substituting (2.4), (2.6) and (2.7) into (3.2) and using the balance principle for the algebraic equation, we find the relationship as

$$\theta = \in +2.$$

Case 1. Choosing $\in = 0$, $\theta = 2$ and $\delta = 1$, we deduce

$$\begin{aligned} (y')^2 &= \frac{{\tau_1}^2(\xi_2\kappa^2 + \xi_1\kappa + \xi_0)}{\zeta_0}, \\ y'' &= \frac{\tau_1(2\xi_2\kappa + \xi_1)}{2\zeta_0}, \end{aligned} \tag{3.3}$$

where $\xi_2 \neq 0$ and $\zeta_0 \neq 0$. By solving the system of equations (2.9), we get

$$\begin{aligned} &w = -(\tau_0 + \frac{\xi_2{\tau_0}^2 - \xi_0\tau_1}{2k\xi_2}),\ \xi_1 = \frac{2\xi_2(k+\tau_0)}{\tau_1},\ \zeta_0 = \xi_2, \\ &\xi_0 = \xi_0,\ \xi_2 = \xi_2,\ \tau_0 = \tau_0,\ \tau_1 = \tau_1. \end{aligned} \tag{3.4}$$

Putting (3.4) into (2.5) and (2.10), we obtain

$$\pm(\eta - \eta_0) = W \int \frac{\mathrm{d}\kappa}{\sqrt{\Lambda(\kappa)}}, \tag{3.5}$$

where

$$\Lambda(\kappa) = \kappa^2 + \frac{\xi_1}{\xi_2}\kappa + \frac{\xi_0}{\xi_2}, W = \sqrt{\frac{\zeta_0}{\xi_2}}. \tag{3.6}$$

Consequently, we write (3.5) as follows

$$\pm(\eta-\eta_0)=\sqrt{\frac{\zeta_0}{\xi_2}}\int\frac{\mathrm{d}\kappa}{\sqrt{\kappa^2+\frac{\xi_1}{\xi_2}\kappa+\frac{\xi_0}{\xi_2}}}=\int\frac{\mathrm{d}\kappa}{\sqrt{\kappa^2+\frac{2(k+\tau_0)}{\tau_1}\kappa+\frac{\xi_0}{\xi_2}}}. \tag{3.7}$$

Integrating (3.7), we obtain the solutions to (3.1) as follows

$$\begin{aligned}\pm(\eta-\eta_0)&=\ln|\kappa-\alpha_1|\\ \pm(\eta-\eta_0)&=2\ln\left|\sqrt{\kappa-\alpha_1}+\sqrt{\kappa-\alpha_2}\right|,\end{aligned} \tag{3.8}$$

where α_1 and α_2 are the roots of $\Lambda(\kappa)$. Substituting the solutions (3.8) into (2.2), we get

$$\begin{aligned}y_1(x,t)&=\tau_0+\tau_1(e^{(x+(\tau_0+\frac{\xi_2{\tau_0}^2-\xi_0\tau_1}{2k\xi_2})t-\eta_0)}+\alpha_1),\\ y_2(x,t)&=\tau_0+\tau_1((e^{(x+(\tau_0+\frac{\xi_2{\tau_0}^2-\xi_0\tau_1}{2k\xi_2})t-\eta_0)}\\ &\quad+(\alpha_1-\alpha_2)^2e^{-(x+(\tau_0+\frac{\xi_2{\tau_0}^2-\xi_0\tau_1}{2k\xi_2})t-\eta_0)}+2(\alpha_1+\alpha_2))(4)^{-1}.\end{aligned} \tag{3.9}$$

If we take $\eta_0=\alpha_2=0$ and $\alpha_1=1$, then we rewrite the solutions (3.9) as single kink solution and the hyperbolic function solution, respectively, as follows

$$\begin{aligned}y_1(x,t)&=\tau_0+\tau_1(e^{(x+A_1t)}+1),\\ y_2(x,t)&=\tau_0+\tau_1\frac{1}{2}(\cosh(x+A_1t)+1),\end{aligned} \tag{3.10}$$

where $A_1=\tau_0+\frac{\xi_2{\tau_0}^2-\xi_0\tau_1}{2k\xi_2}$ and the inverse width of solitons is 1.
Case 2. Now choosing $\in=0,\ \theta=2$ and $\delta=2$, we get

$$\begin{aligned}y&=\tau_0+\tau_1\kappa+\tau_2\kappa^2,\ (y')^2=\frac{(\tau_1+2\tau_2\kappa)^2(\xi_2\kappa^2+\xi_1\kappa+\xi_0)}{\zeta_0},\\ y''&=\frac{4\tau_2(\xi_2\kappa^2+\xi_1\kappa+\xi_0)+(\tau_1+2\tau_2\kappa)(2\xi_2\kappa+\xi_1)}{2\zeta_0},\end{aligned} \tag{3.11}$$

where $\xi_2\neq0,\ \zeta_0\neq0$. By solving the systems of algebraic equation (2.9), we obtain

$$\begin{aligned}&\xi_0=\xi_0,\ \xi_1=\frac{\xi_2\tau_1}{\tau_2},\ \xi_2=\xi_2,\ \zeta_0=4\xi_2,\\ &\tau_0=\frac{{\tau_1}^2-8k\tau_2}{8\tau_2},\ \tau_1=\tau_1,\ \tau_2=\tau_2,\ w=\frac{-64k^2{\tau_2}^2+{\tau_1}^4}{128k{\tau_2}^2}.\end{aligned} \tag{3.12}$$

Substituting these results into (2.5), we have

$$\pm(\eta-\eta_0)=\sqrt{\frac{\zeta_0}{\xi_2}}\int\frac{\mathrm{d}\kappa}{\sqrt{\kappa^2+\frac{\xi_1}{\xi_2}\kappa+\frac{\xi_0}{\xi_2}}}=2\int\frac{\mathrm{d}\kappa}{\sqrt{\kappa^2+\frac{\tau_1}{\tau_2}\kappa+\frac{\xi_0}{\xi_2}}}. \tag{3.13}$$

Integrating (3.13), we obtain the solutions to (3.1) as follows

$$\begin{aligned} \pm(\eta-\eta_0) &= 2\ln|\kappa-\alpha_1|, \\ \pm(\eta-\eta_0) &= 4\ln\left|\sqrt{\kappa-\alpha_1}+\sqrt{\kappa-\alpha_2}\right|, \end{aligned} \quad (3.14)$$

where α_1 and α_2 are the roots of $\Lambda(\kappa)$. Substituting the solutions (3.14) into (2.2), we get

$$\begin{aligned} y_3(x,t) &= \tau_0+\tau_1(e^{\frac{1}{2}(x-\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}t-\eta_0)}+\alpha_1)+\tau_2(e^{\frac{1}{2}(x-\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}t-\eta_0)}+\alpha_1)^2, \\ y_4(x,t) &= \tau_0+\tau_1((e^{\frac{1}{2}(x-\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}t-\eta_0)}+(\alpha_1-\alpha_2)^2e^{-\frac{1}{2}(x-\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}t-\eta_0)}+2(\alpha_1+\alpha_2))(4)^{-1}) \\ &+\tau_2((e^{\frac{1}{2}(x-\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}t-\eta_0)}+(\alpha_1-\alpha_2)^2e^{-\frac{1}{2}(x-\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}t-\eta_0)}+2(\alpha_1+\alpha_2))(4)^{-1})^2. \end{aligned} \quad (3.15)$$

For simplicity, $\eta_0=\alpha_2=0$ and $\alpha_1=1$, then we rewrite the solutions (3.15) as

$$\begin{aligned} y_3(x,t) &= \left[\sum_{j=0}^{2}\tau_j\left(e^{\frac{1}{2}(x-A_2t)}+1\right)^j\right], \\ y_4(x,t) &= \left[\sum_{j=0}^{2}\tau_j\left(\frac{1}{4}\left(\cosh\left[(x-A_2t)\right]+1\right)\right)^j\right], \end{aligned} \quad (3.16)$$

where $A_2=\frac{-64k^2\tau_2{}^2+\tau_1{}^4}{128k\tau_2{}^2}$ and the inverse width of solitons is $\frac{1}{2}$.

4 Conclusion

In this work, we obtain the hyperpolic function and single kink solution of the Camassa-Holm equation using the ETEM. The performance of method provide a remarkable and impressive result for the problem under consideration. Consequently, we observe that ETEM contains an effective algorithm and powerful solution for the model problem.

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